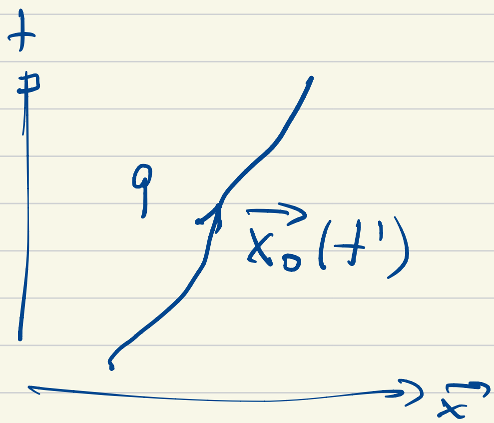
Lecture 6

- Radiation of a moving charge (cont.)
  - physical properties
- Multipole expansion
  - Electrostatics
  - Multipole moments
  - Magnetostatics

From Lecture 5:



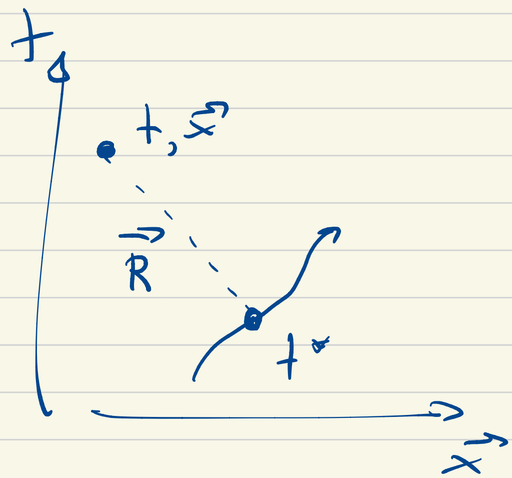
$$\vec{E}(\vec{x}) =$$

$$= \frac{q}{4\pi\varepsilon_0} \frac{1}{(1-\vec{\beta}\cdot\vec{n})^3} \frac{1}{cR} \vec{n} \times \left([\vec{n} - \vec{\beta}] \times \dot{\vec{\beta}} \right) \Big|_{\text{all at } t_*} + O\left(\frac{1}{R^2}\right)$$

$$\vec{R} = \vec{x} - \vec{x}_0(t_*)$$

$t_*(t, \vec{x})$:

$$c(t - t_*) = R(t_*)$$



$$\vec{B} = \frac{1}{c}[\vec{n} \times \vec{R}]$$

$$\frac{d\mathcal{E}}{d\Omega\, dt} = \lim_{R\to\infty} R^2 \vec{S}\cdot\vec{n}$$

$$\vec{S} = \frac{1}{\mu_0}[\vec{E} \times \vec{B}]$$

moving charge : $|R - \vec{\beta} \cdot \vec{R}|$ → is the distance to the charge at moment t (for constant velocity).

various β - dependent factors come from "changing the frame".

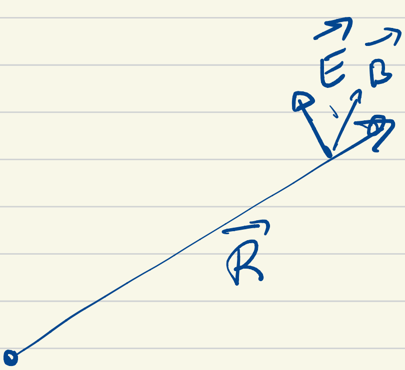
- The $\frac{1}{R}$ term is the radiation

$$\frac{q}{4\pi\varepsilon_0} \frac{1}{(1-\vec{\beta}\cdot\vec{n})^3} \frac{1}{cR} \vec{n} \times [\vec{n} - \vec{\beta}] \times \dot{\vec{\beta}}$$

the energy flux:

$$\frac{d\mathcal{E}}{d\Omega dt} = \lim_{R\to\infty} R^2 \vec{S} \cdot \vec{n} \;, \qquad \vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

$$\frac{1}{\mu_0} \left(\vec{E} \times \vec{n} \times \vec{E} \right) \cdot \vec{n} = \frac{E^2}{\mu_0}$$



$$\frac{1}{c\mu_0} \lim_{R\to\infty} R^2 |\vec{E}|^2 =$$

$$= \frac{q^2}{16\pi^2 \underbrace{\varepsilon_0^2 \cancel{\mu_0} c^3}_{\varepsilon_0 c}} \frac{\left(\vec{n} \times (\vec{n} - \vec{\beta}) \times \dot{\vec{\beta}}\right)^2}{(1 - \vec{n}\cdot\vec{\beta})^6}$$

• Non-relativistic limit:

$$|\vec{\beta}| \ll 1 \qquad \left[\vec{\beta} = \frac{\vec{v}}{c} \right] \qquad \dot{\vec{\beta}}$$

$$\frac{d\mathcal{E}}{d\Omega\, dt} \simeq \frac{q^2}{16\pi^2 \varepsilon_0 c} \left| \vec{n} \times \dot{\vec{\beta}} \right|^2$$

$$\frac{dP}{dt} = \int d\Omega \sin^2\theta \frac{q^2}{16\pi^2 \varepsilon_0 c} |\dot{\vec{\beta}}|^2 =$$

$$\int_0^{\pi} d\varphi \sin\theta \, d\theta$$

$$\left[\int_0^{\pi} \sin^3\theta = \frac{4}{3} \right] \qquad = \frac{q^2 |\dot{\vec{\beta}}|^2}{6\pi \varepsilon_0 c}$$

Larmor Formula

In the relativistic case it is important to distinguish the radiation emitted in the frame of the particle:

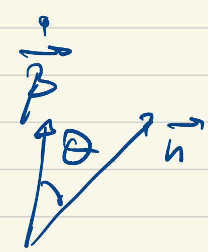
$$\frac{d\mathcal{E}}{d\Omega\, dt'} = \frac{d\mathcal{E}}{d\Omega\, dt} \cdot \frac{dt}{dt'} \qquad \frac{dt'}{dt} = \frac{1}{1-\vec{\beta}\cdot\vec{n}}$$

$$|\vec{\beta}| \simeq 1 \qquad 1-\vec{\beta}\cdot\vec{n} \longrightarrow 0$$

$$\vec{\beta} \parallel \vec{n}$$

- Rectilinear motion:

$$\beta \parallel \dot{\beta} \quad , \quad \frac{q^2}{16\pi^2 \varepsilon_0 c} \frac{\left(\vec{n}\times(\vec{n}-\vec{\beta})\times\dot{\vec{\beta}}\right)^2}{(1-\vec{n}\cdot\vec{\beta})^6}$$



$$\frac{d\mathcal{E}}{d\Omega\, dt'} = \frac{q^2}{16\pi^2 \varepsilon_0 c} \frac{\dot{\vec{\beta}}^2 \sin^2\Theta}{\left(1-\beta\cos\Theta\right)^5}$$

total power: $\dfrac{d\mathcal{E}}{dt'} = \dfrac{q^2 |\dot{\beta}|^2}{6\pi \varepsilon_0 c} \gamma^6 \quad , \quad \gamma = \dfrac{1}{\sqrt{1-\beta^2}}$

Derivation:

$$\int_0^{\pi} \frac{d\theta' \sin^3\theta}{(1-\beta\cos\theta)^5} = \int_{-1}^{1} \frac{dx\,(1-x^2)}{(1-\beta x)^5} = \frac{4}{3}\frac{1}{(1-\beta^2)^3}.$$

Consider

$\beta \to 1$. $\gamma \to \infty$, all radiation near $\theta \sim 0$

$$\gamma \sim \frac{1}{\sqrt{2(1-\beta)}} \quad : \quad 2\gamma^2 \sim \frac{1}{1-\beta}$$

$$\frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \longrightarrow \frac{\theta^2}{\left(1-\beta\left(1-\frac{\theta^2}{2}\right)\right)^5} =$$

$$= \frac{\theta^2}{\left(\frac{1}{2\gamma^2}+\frac{\theta^2}{2}\right)^5} = \frac{2^5\gamma^{10}\theta^2}{(1+\gamma^2\theta^2)^5}$$

$$\frac{dE}{d\Omega\,dt'} = \frac{2q^2}{\pi^2\varepsilon_0 c}|\dot{\vec{\beta}}|^2\gamma^8\frac{(\gamma\theta)^2}{(1+(\theta\gamma)^2)^5}$$

$(\theta\gamma)^2 = \frac{1}{4} \to$ max.

Multipole Expansion

- Review of multi-dimensional Taylor formula

two variables:

$$f(x,y) =$$

$$\sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{m!}\frac{1}{k!} x^m y^k \partial_x^m \partial_y^k f\Big|_{x,y=0} =$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{(n-k)!}\frac{1}{k!} x^{n-k} y^k \partial_x^{n-k} \partial_y^k f\Big|_{x,y=0} =$$

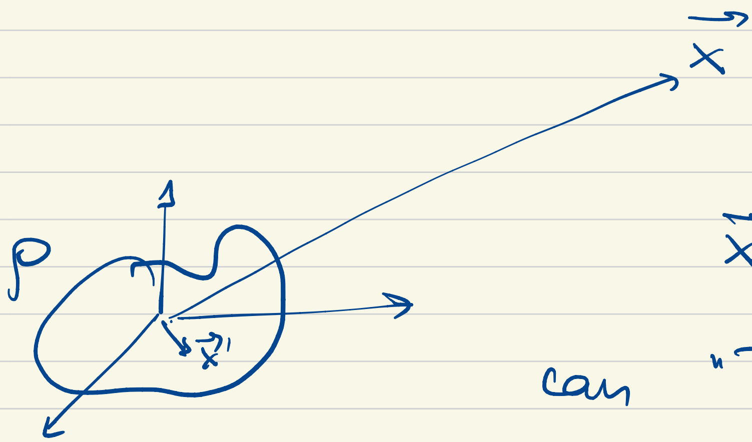
$$= \sum_{n=0}^{\infty} \underbrace{\sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \cdots \sum_{i_n=1}^{2}}_{2^n \text{ terms}} \frac{x_{i_1} \cdots x_{i_n}}{n!} \partial_{x_{i_1}} \cdots \partial_{x_{i_n}} f$$

2^n terms $(x_1 \equiv x, \; x_2 \equiv y)$

$\frac{n!}{(n-k)!\,k!}$ have k ∂_y derivatives

Any dimension:

$$f(\vec{x}) = \sum_{n=0}^{\infty} \frac{\overbrace{x_{i_1} \cdots x_{i_n}}^{n\text{-tensor}}}{n!} \overbrace{\partial x_{i_1} \cdots \partial x_{i_n}}^{n\text{-tensor}} f\Big|_{\vec{x}=0}$$



$\vec{x} >> \vec{x}' \longrightarrow$ we can "Taylor expand" the internal structure.

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3\vec{x}' \, \frac{\rho(x')}{|\vec{x} - \vec{x}'|}$$

Let us start by Taylor expanding the Green's function:

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{n=0}^{\infty} \frac{1}{n!} x'_{i_1} \cdots x'_{i_n} \frac{\partial}{\partial x'_{i_1}} \cdots \frac{\partial}{\partial x'_{i_n}} \frac{1}{|\vec{x} - \vec{x}'|}\Bigg|_{\vec{x}'=0} =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{x'_{i_1} \cdots x'_{i_n}}{|\vec{x}|^{2n+1}} T_{i_1 \dots i_n}(\vec{x})$$

$$T_{i_1 \dots i_n}(\vec{x}) = |\vec{x}|^{2n+1} \frac{\partial}{\partial x'_{i_1}} \cdots \frac{\partial}{\partial x'_{i_n}} \frac{1}{|\vec{x} - \vec{x}'|}\Bigg|_{\vec{x}'=0}$$

T is symmetric and traceless:

$$T_{ii \dots i_n} \approx \Delta_{x'} \frac{1}{|\vec{x} - \vec{x}'|} \approx \delta(\vec{x} - \vec{x}') = 0$$

$$(\vec{x} >> \vec{x}')$$

$$T_{i_1 \dots i_n}(\vec{x}) = (2n-1)!! \, x_{i_1} x_{i_2} \cdots x_{i_n} -$$

$$- A_{i_1 \dots i_n},$$

where tensor A contains at least one kronecker δ_{ij}

$$T_i = x_i$$

$$T_{ij} = 3 x_i x_j - x^2 \delta_{ij}$$

• Stopped here

We would like to expand the expression

$$\int d^3\vec{x}' \frac{\rho(x')}{|\vec{x}-\vec{x}'|} \quad \text{at large} \quad \frac{|\vec{x}|}{|\vec{x}'|}$$

But so far it is not clear which integrals of $\int \rho(\vec{x}')\, x'_{i_1} \ldots x'_{i_n}$ will or will not contribute because of the structure of $T(\vec{x})$.

We then use the following trick:

$$x'_{i_1} x'_{i_2} \ldots x'_{i_n} T_{i_1 \ldots i_n}(\vec{x}) =$$

$A \sim \delta_{ij}$

$\ldots \delta_{ij} \ldots T_{\ldots ij \ldots} = 0$

$$= \left(x'_{i_1} x'_{i_2} \ldots x'_{i_n} - \frac{A_{i_1 \ldots i_n}(\vec{x}')}{(2n-1)!!} \right) T_{i_1 \ldots i_n}(\vec{x}) =$$

$$= \frac{P_{i_1 \ldots i_n}(\vec{x}')}{(2n-1)!!} \left(T_{i_1 \ldots i_n}(\vec{x}) + A_{i_1 \ldots i_n}(\vec{x}) \right)$$

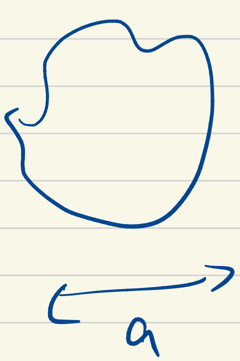
$$= P_{i_1 \ldots i_n}(\vec{x}\,') \, x_{i_1} \ldots x_{i_n}$$

$$\varphi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3\vec{x}\,' \rho(x') \times$$

$$\times \sum_{n=0}^{\infty} \frac{1}{n!} \frac{x'_{i_1} \ldots x'_{i_n}}{|\vec{x}|^{2n+1}} T_{i_1 \ldots i_n}(\vec{x}) =$$

$$= \frac{1}{4\pi\varepsilon_0} \int d^3\vec{x}\,' \sum_{n=0}^{\infty} \frac{1}{n!} \rho(x') T_{i_1 \ldots i_n}(\vec{x}\,') \frac{x_{i_1} \ldots x_{i_n}}{|\vec{x}|^{2n+1}} =$$

$$= \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{1}{n!} Q_{i_1 \ldots i_n} \frac{x_{i_1} \ldots x_{i_n}}{|\vec{x}|^{2n+1}}$$



$\Rightarrow Q_{i_1 \ldots i_n} \sim q\, a^n$ $\left[\text{since } \int_0^a dx'\, x'^n \sim a^n\right]$

n_{th} term goes as $\dfrac{q}{|\vec{x}|} \left(\dfrac{a}{|\vec{x}|}\right)^n$

Let us derive the corresponding electric fields:

$$\varphi^{(0)} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r} \quad \Rightarrow \quad E^{(0)} = \frac{q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3}$$

$$\varphi^{(1)} = \frac{1}{4\pi\varepsilon_0}\frac{\vec{Q}^{(1)}\cdot\vec{x}}{|x|^3} \quad \Rightarrow$$

$$\vec{E} = -\vec{\nabla}\varphi$$

$$\vec{E}^{(1)} = \frac{1}{4\pi\varepsilon_0}\left(-\frac{\vec{Q}^{(1)}}{|x|^3} + \frac{3\vec{x}\,(\vec{Q}^{(1)}\cdot\vec{x})}{|x|^5}\right)$$

Magnetostatics

Since the Green's function for the vector potential is very similar we can use the similar techniques:

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

We will not derive the all-orders expression, but just work at the leading orders:

$$A_i \approx \frac{\mu_0}{4\pi} \frac{1}{x} \int d^3x' J_i(x') +$$

$$+ \frac{x_j}{|\vec{x}|^3} \int d^3x' J_i(x') x'_j$$

Charge conservation: $\vec{\nabla} \cdot \vec{J} + \cancel{\dot{\rho}}^{\,0} = 0 \Rightarrow$

$$\vec{\nabla} \cdot \vec{J} = 0$$

$$\int_V x_i \partial_j J_j = 0 \Rightarrow \int_V \partial_j (x_i J_j) -$$

$$- \int_V J_i = 0 \quad \Rightarrow \quad \int_V J_i = 0$$

No magnetic monopoles ! $\vec{\nabla} \cdot \vec{B} = 0$

We can also show that

$$\int_V d^3x \, (x_i J_j + x_j J_i) = 0$$

Dipole term:

$$x_j \int d^3x' \, J_i(x') \, x'_j =$$

$$= \frac{1}{2} x_j \int d^3x' \left(J_i(x') \, x'_j - J_j(x') \, x'_i \right) =$$

$$= \frac{1}{2} \int d^3x' \left(\vec{J} \, (\vec{x} \cdot \vec{x}') - \vec{x}' (\vec{J} \cdot \vec{x}) \right) =$$

$$\left[\vec{a} \times \vec{b} \times \vec{c} = \vec{b} (\vec{a} \cdot \vec{c}) - \vec{c} (\vec{a} \cdot \vec{b}) \right]$$

$$= \frac{1}{2}\int d^3x' \, \vec{x} \times \vec{J} \times \vec{x'} = \frac{1}{2}\underbrace{\Big(\int d^3x' \, \underbrace{\vec{x'} \times \vec{J}(x')}_{\vec{M}(x')}\Big)}_{\vec{m}} \times \vec{x}$$

$\vec{M}(x') \sim$ magnetisation

$\vec{m} \sim$ magnetic moment

$$\vec{A}^{(1)} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{x}}{|\vec{x}|^3} + \mathcal{O}\left(|x|^{-3}\right)$$

$$\vec{B}^{(1)} = \vec{\nabla} \times \vec{A}^{(1)} = \ldots = \frac{\mu_0}{4\pi} \cdot \frac{3\vec{n}\,(\vec{m}\cdot\vec{n}) - \vec{m}}{|\vec{x}|^3}$$

Note the similarity:

$$\vec{E}^{(1)} = \frac{1}{4\pi\varepsilon_0}\left(\frac{3\vec{n}\,(\vec{Q}^{(1)}\cdot\vec{n}) - \vec{Q}^{(1)}}{|\vec{x}|^3}\right)$$